Application of the Buckingham Pi Theorem to Dam Breach Equations

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Abstract

Before the recent collapse of a major corporation, a Fortune magazine journalist asked a simple question, "How do you make money?" In this business there is an equally simple question, "How do you do math?" Sometimes an assemblage of dimensionless variables is presented as a case of dimensional analysis, misapplying the Buckingham Pi theorem. The theorem, its usage and its limitations are reviewed in the context of water flow through a dam breach model, taken from a set of measurements of flow through trapezoidal notch in a trapezoidal reservoir embankment made of plywood in a flume.

Keywords

Buckingham Pi theorem, dam breach, head-discharge relation

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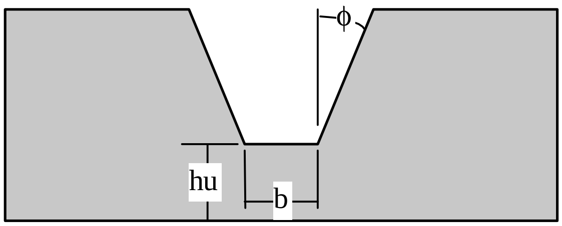
Technical notes

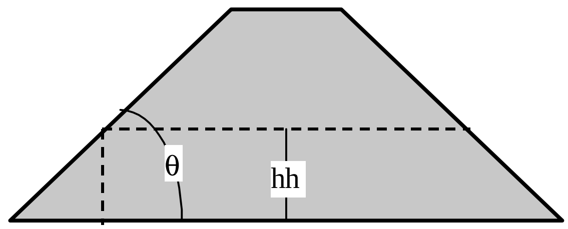
The Buckingham Pi theorem may sometimes be misused as a general solution method for complex engineering problems. In his text, Applied Mathematics, Logan (1987) gives the example of its application to the expansion of the fireball of a nuclear explosion. In this case, there are four pertinent physical quantities expressed in three physical dimensions. According to the theorem, this leaves one possible dimensionless pi-variable to express the physical law of expansion, from which one can obtain $r^5p/(t^2e)$ = constant, where r is the radius of the fireball, p is the atmospheric pressure outside the fireball, t is the time and e is the energy released in the blast. In this special case, the basic physical law, confirmed by later experiments, literally falls out of the non-dimensional analysis.

But it goes too far to say that all physical laws can fall out of non-dimensional analysis. Logan notes, “the existence of a physical law is an assumption. In practice one must conjecture which are the relevant variables in a problem and then apply the machinery of the theorem. The resulting dimensionless physical law must be checked by experiment, or whatever, in an effort to determine the validity of the original assumptions.” Nor is it necessarily valid to take any particular pi-variable and construct a physical law by setting that variable equal to the general formula for pi-variables in a given problem.

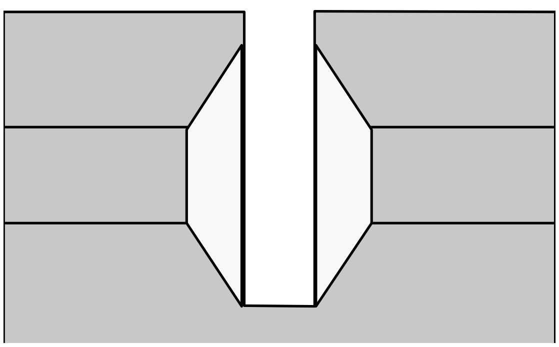
Take for example the case of water flowing in a trapezoidal notch through the trapezoidal embankment of a reservoir. The notch has a bottom width, b (units of length, L), at height, hu (L), above the reservoir floor, with a sidewall slope angle of ϕ (rad) from the vertical. The embankment has an upstream slope angle of θ (rad) above the horizontal. The intersection of the notch with the upstream slope is called the crest of the resulting flume or weir. In addition, the floor of the notch downstream of the crest can be drop, hh (L), below the crest, forming a rectangular channel below the crest, so that the water flowing into it can be an aerated waterfall, or nappe. The pictures below show one such condition with hh = hu in the drawn figures and hh = 0 in the photograph of the plywood model. This model was built at the Hydraulic Engineering Research Unit of the USDA Agricultural Research Service, Stillwater, OK, and was demonstrated on June 27, 2001, to participants of the FEMA sponsored Workshop on Issues, Resolutions, and Research Needs Related to Dam Failure Analyses, Oklahoma City, OK.

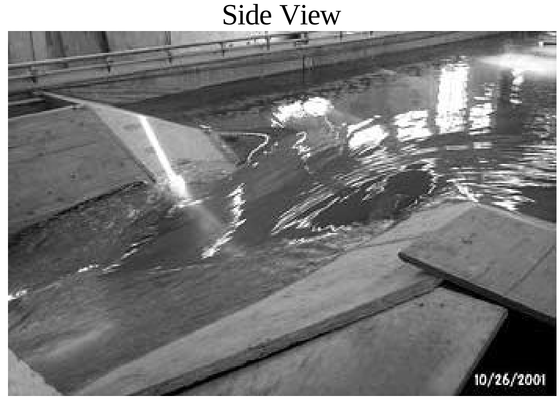


View from Upstream



Side View


Top View



Suppose that one constructs, as did one proposal, a non-dimensional equation (1) to explain the flow, Q, through the notch. Is it a legitimate construction of the Buckingham Pi theorem and does it constitute a valid “physical law”?

$$\frac{Q}{\sqrt{g \bullet he^5}} = C_1 \left(\frac{he}{hu + he} \right)^{C_2} \left(\frac{he}{hh + he} \right)^{C_3} \left(\frac{b}{he} \right)^{C_4} \cos^{C_5}(f) \sin^{C_6}(q) , \qquad (1)$$

where Q (L^3/T) is discharge, g (L/T^2) is the acceleration of gravity, he (L) is the elevation head of the reservoir above the crest behind the influence of the notch, hu, hh f and q are defined as before, and C_1 through C_6 are dimensionless fitting parameters.

According to Logan, the Buckingham Pi theorem says:

Let $f(q_1, q_2, ..., q_m) = 0$ *be a unit-free physical law that relates the dimensional quantities* $q_1, q_2, \ldots, q_m$. *Let* $L_1, \ldots, L_n$ $(n < m)$ *be fundamental dimensions with* $[q_i] = L_1^{a1i} \bullet L_2^{a2i} \bullet\bullet\bullet L_n^{ani}$ *, i = 1, …, m. And let r = rank A, where A is the dimension matrix (2). Then there exists m-r independent dimensionless quantities,* $\boldsymbol{p}_1, \boldsymbol{p}_2, \ldots, \boldsymbol{p}_{m-r}$*, which can be formed from* $q_1, q_2, \ldots, q_m$*, and the physical law is equivalent to an equation* $F(\boldsymbol{p}_1, \boldsymbol{p}_2, ..., \boldsymbol{p}_{m-r}) = 0$ *expressed only in terms of the dimensionless quantities.*

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1m} \\ a_{21} & \cdots & a_{2m} \\ | & & | \\ a_{n1} & \cdots & a_{nm} \end{bmatrix}, \qquad (2)$$

where a_{ij} are the power variables of the dimensions L_j for each q_i.

In many cases, r = n. The term "unit-free physical law" only means that the dimensions of both sides of an equation cancel out, so that one cannot set mass equal to length, for example.

Logan follows the theorem with a diffusion example which we can follow. To determine whether or not sin(θ) and cos(ϕ) are legitimate π-variables, let us define three lengths:

dn = the depth (L) of the notch from the top of the embankment to the horizontal crest

lu = the length (L) of the upstream slope from the crest to the top of the embankment, such that sin(θ) = dn/lu

ls = the length (L) of the inclined side slope of the trapezoidal notch such that cos(ϕ) = dn/ls

We now have nine physical variables, Q, g, he, b, dn, lu, ls, hh and hu, in terms of two dimensions, length (L) and time (T). Following Logan's example, we set up a dimension matrix to determine the power equations necessary to derive the dimensionless π-variables (3).

	Q	g	he	b	dn	lu	ls	hh	hu
L	3	1	1	1	1	1	1	1	1
T	-1	-2	0	0	0	0	0	0	0
power	a1	a2	a3	a4	a5	a6	a7	a8	a9

$\pi_i = Q^{a1}\ g^{a2}\ he^{a3}\ b^{a4}\ dn^{a5}\ lu^{a6}\ ls^{a7}\ hh^{a8}\ hu^{a9}$, where $a1_i, \ldots, a9_i$ are assumed (3)

From (3) we set up the equation for the dimensions of the π-variables (4). This leads to the equations (5) and (6), which must be satisfied simultaneously to produce dimensionless π-variables. Equation (5) can be satisfied only by a1 = a2 = 0 or a2 = -a1/2. If the second choice is taken, and we set a1 = 1, then equation (6) becomes (7), and a2 = -1/2. We can set a4 = … = a9 = 0, which with (7) produces the (8) and the left hand side of (1), which is a valid π-variable, π1.

$$[\pi_i] = T^{(-a1-2a2)}\ L^{3a1+a2+a3+a4+a5+a6+a7+a8+a9} \quad (4)$$

$$a1 + 2\ a2 = 0 \quad (5)$$

$$3\ a1 + a2 + a3 + a4 + a5 + a6 + a7 + a8 + a9 = 0 \quad (6)$$

$$(5/2)a1 + a3 + a4 + a5 + a6 + a7 + a8 + a9 = 0 \quad (7)$$

$$\pi_1 = Q/(g\ he^5)^{1/2} \quad (8)$$

It is obvious from (4), (5) and (6) that the rank of the A matrix is r = 2. We have m = 9, so we need m - r = 7 π-variables to completely specify the problem in dimensional analysis. But equation (1) only has 6 dimensionless variables. Suppose that we set a1 = a2 = 0, and work with (6) to get the remaining 6 π-variables. It can be shown that the following solutions form an independent set of 7 with π1:

π2 = b/he, π3 = dn/lu = sin(θ), π4 = dn/ls = cos(ϕ), π5 = he/hh, π6 = he/hu, π7 = he/dn

But there is no way with this method to generate the dimensionless variables (he/(hu+he)) or (he/(hh+he)).

So equation (1) may be missing one π-variable and has two that are not valid. It is surely constructed of dimensionless variables, but it does not satisfy the Buckingham Pi theorem. Therefore it does not strictly illustrate an example of dimensional analysis. One should note that it may be possible to argue that when he < dn, the depth of the notch, dn, should have no effect on the flow. Therefore, one could dispense with π7. But if he > dn, then the flow overtops the embankment and π7 becomes necessary, as may other variables.

Nor does the Buckingham Pi theorem guarantee that any collection of legitimate π-variables would produce a valid head-discharge relation. The theorem only guarantees that if a physical law exists, then the terms can be rearranged into m - r dimensionless π-variables. One can see this by considering a known special case solution to the current problem (Fread, 1991). If the downstream floor of the notch is level with the horizontal crest, hh = 0, then the broad-crested weir equation (9) applies, where the dimensions of the coefficients 3 and 2 are the same as the square root of gravity, g (L/T^2).

$$Q = c0\ b\ he^{3/2} + c1\ \tan(\phi)\ he^{5/2}\text{, where } c0 = 3\ (L^{1/2}/T) \text{ and } c1 = 2\ (L^{1/2}/T) \qquad (9)$$

The $he^{3/2}$ term comes from flow over the horizontal floor of the notch and the $he^{5/2}$ term comes from flow over the inclined sides of the notch. We can consider the tan(ϕ) term to be a dimensionless coefficient and leave it out of the dimensional analysis on the basis that he < dn. Notice that equation (9) has two terms while equation (1) only has one. Since (9) has been confirmed by both theory and experiment, one can take the five physical variables, Q, b, he, c0 and c1, and construct a valid dimensionless equation with m – r = 3 π-variables, such as:

$$p_2 + p_1 p_3 = p_1 p_2 p_3 \text{ , where } p_1 = \frac{Q}{c0 \bullet he^{5/2}}, p_2 = \frac{Q}{c1 \bullet \tan(f) \bullet he^{5/2}}, p_3 = \frac{he}{b} \qquad (10)$$

However, as (10) demonstrates, merely having a dimensionless equation does not guarantee any greater understanding. If an equation like (9) is correct and works properly at different physical scales, an equation like (10) is no better or worse. It is only a rearrangement of the terms. If (1), or any mathematically equivalent variant with dimensioned variables, cannot fit a known special case like (9), then it cannot be true in general.

Dimensional analysis can not necessarily solve general problems, nor can it even add to understanding unless it is used to some legitimate purpose. In his exposition on subcritical and supercritical flow in rectangular channels, Henderson (1966) uses a dimensionless equation to collapse a family of curves of depth, y (L), versus specific energy, E (L) into a single line. Science is replete with examples where dimensionless variables make integrals go neatly from zero to one.

In this case at least one application of the Buckingham Pi theorem may be useful as a means of distinguishing between flow regimes. Consider the ideal weir equation (11) for an aerated jet over a sharp crested weir.

$$dQ = C \bullet \frac{2\sqrt{2 \bullet g}}{3} h_e^{3/2} \bullet ds \text{ ,} \qquad (11)$$

where Q(L^3/T) is flow, C is a dimensionless coefficient of discharge that accounts for inertia and shear losses, g (L/T^2) is the acceleration of gravity, he (L) is reservoir the entrance head well behind the influence of the weir, and s (L) is the weir length.

This equation (John and Haberman, 1988; Gupta, 1989) assumes that the flow vector over a weir is perpendicular to the weir and the pressure distribution vertically through the aerated jet is hydrostatic. It ignores secondary effects, such as contraction coefficients and velocity of approach. It is effectively identical in form to the equation for flow over a broad crested weir.

If one uses it on the model dam breach as a broad crested weir, where hh = 0 and ds is integrated directly across the notch, equation (9) re-emerges as (12), where b0 and b1 are dimensionless discharge coefficients. If one assumes the jet past the entrance is fully aerated and integrates along the horizontal crest of the entrance and up the inclined edges where the upstream slope of the dam intersects with the side slopes of the notch, one gets (13), where a0 and a1 are dimensionless discharge coefficients. If one folds the number fractions into dimensionless coefficients, c0 and c1, and the trigonometric functions into the second dimensionless discharge coefficient, c1, both forms become as (14).

$$Qb = b0 \bullet \frac{2\sqrt{2g}}{3} \bullet b \bullet he^{3/2} + b1 \bullet \frac{8\sqrt{2g}}{15} \bullet \tan(f) \bullet he^{5/2} \tag{12}$$

$$Qa = a0 \bullet \frac{2\sqrt{2g}}{3} \bullet b \bullet he^{3/2} + a1 \bullet \frac{8\sqrt{2g}}{15} \bullet \left(\cot^2(q) + \tan^2(f)\right)^{1/2} \bullet he^{5/2} \tag{13}$$

$$Q = c0 \bullet \sqrt{g} \bullet b \bullet he^{3/2} + c1 \bullet \sqrt{g} \bullet he^{5/2} \tag{14}$$

The remaining physical variables with dimensions are, Q (L^3/T), g (L/T^2), b (L) and he (L), in terms of length (L) and time (T). According to the Buckingham Pi theorem, this allows for two independent π-variables, with which (14) can be rearranged into (15).

$$p_q = c0 + c1 \bullet p_e \text{ , where } p_q = \frac{Q}{\sqrt{g \bullet b^2 \bullet he^3}} \text{ and } p_e = \frac{he}{b} \tag{15}$$

Of course, c0 and c1 can be functions of the other unstated π-variables, such as cot(θ), tan(ϕ), hh/b and hu/b. If one assumes that c0 and c1 have no dependence on π_e or he, then (15) defines a straight line in $\pi_q(\pi_e)$ for a given geometry. If the measured data, (π_e, π_q), do not form a straight line, the two things are possible. Either the data is bad, or the assumptions do not hold. The assumptions may not hold if the flow is affected by entrance velocity, contractions or de-aeration of the jet. That makes this form useful for determining the intrusion of different flow regimes.

Equation (15) encourages the use of least squares (Press, et al., 1989) to fit measured (π_e, π_q) data. One should also note that this may produce somewhat different results than if one uses simulated annealing (Goffe, et al., 1994) with the mean relative absolute error (16) to fit equations (12), (13) or (14) directly.

$$f = \frac{1}{n}\sum_{i=1}^{n} \frac{\left| Q_i^{\wedge} - Q_i \right|}{\sqrt{Q_i^{\wedge} \bullet Q_i}} \tag{16}$$

Conclusions

Except in the special condition where the number of dimensioned physical variables exceeds the number of physical dimensions by one, the Buckingham Pi theorem will not necessarily generate an unknown physical law, especially for complex problems. It is merely a rearrangement of terms of a known equation into the minimum possible number of dimensionless variables. Therefore, the practice of assembling a product of powers of plausible dimensionless variables into an equation has no particular physical or mathematical basis for producing valid equations to fit experimental data. Nor are all applications of the Buckingham Pi theorem to know relations necessarily insightful. But in one example the Buckingham Pi theorem can be used to convert a highly nonlinear representation of the ideal weir equation, applied to a physical dam breach model by integration, to a linear equation in π-variables. This may have the advantage of making the changes to other flow regimes more visible as deviations from straight lines.

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