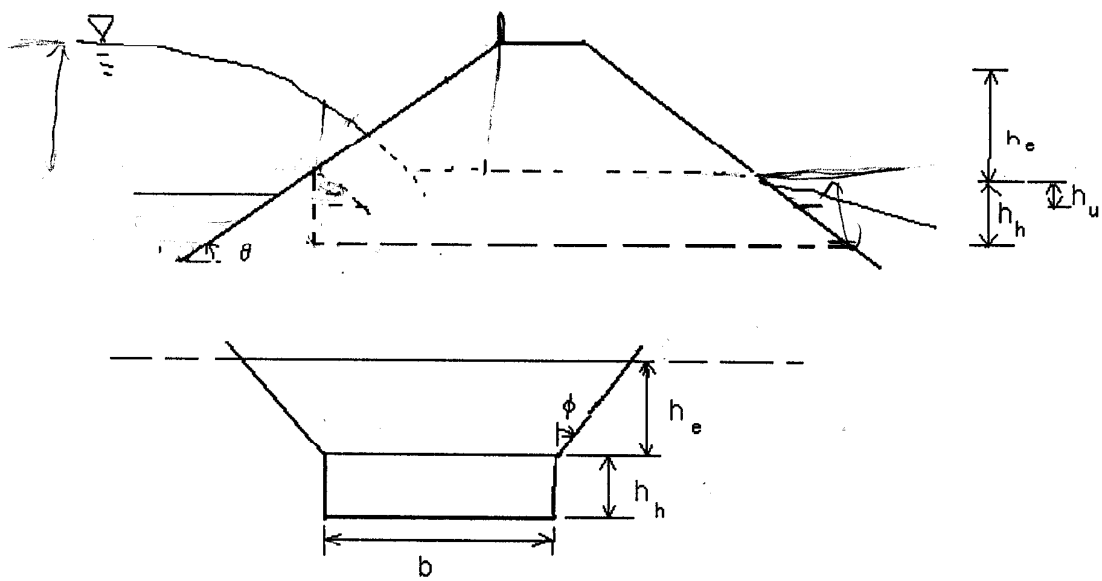
RELATION DERIVATION
DAM BREACH HEAD-DISCHARGE STUDY

(Revised 4-30-99)

APPROACH 1: Dimensional Analysis to look at relational forms.

With fluid properties assumed constant, the variables are: Q, g, h_e, h_h, h_u, b, θ, and ϕ. Defined as (see sketch):

- Q = volumetric discharge through the breach (L^3T^{-1}),
- g = gravitational acceleration (LT^{-2}),
- h_e= the head above the brink of the headcut (L),
- h_h= the height of the headcut (L),
- h_u= the height of the headcut above the upstream bed (L),
- b = the width of the headcut at the overfall (L)
- θ = the angle of the upstream embankment slope with the horizontal (radians: dimensionless)
- ϕ= the angle of the breach side slope with the vertical (radians: dimensionless)



8 variables in two dimensions implies 6 dimensionless terms. One formulation of these terms yields the relation:

$$\frac{Q}{\sqrt{gh_e^5}} = f\left(\frac{h_u}{h_e}, \frac{h_h}{h_e}, \frac{b}{h_e}, \theta, \phi\right) \qquad (1)$$

where the function on the right hand side of the equation may be formulated in various ways. One approach is to combine the dimensionless quantities on the right side of equation 1 into terms lie on the interval (0,1] and assume a simple power form of the function. A possible formulation of this approach is:

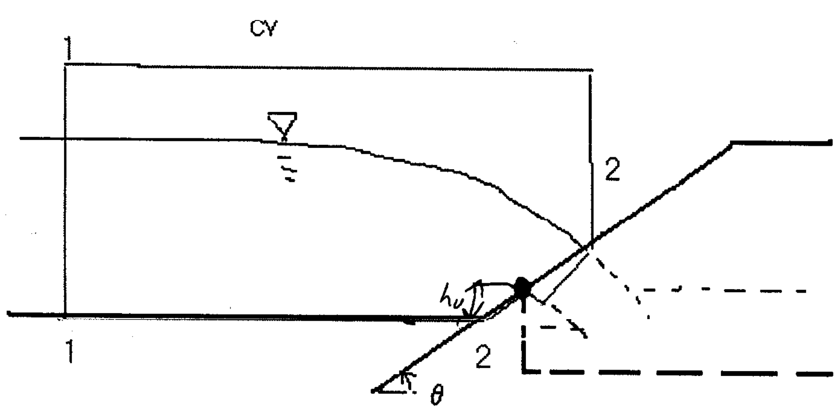
$$\frac{Q}{\sqrt{gh_e^5}} = C_1\left(\frac{h_e}{h_u + h_e}\right)^{C_2}\left(\frac{h_e}{h_h + h_e}\right)^{C_3}\left(\frac{b}{h_e}\right)^{C_4} \cos^{C_5}(\phi)\sin^{C_6}(\theta) \qquad \text{(1a)}$$

Note that although the formulation is arbitrary, it is made with rational consideration of the anticipated influence of the various parameters. For example, the headcut base width, b, is not forced to the interval (0,1] with the other parameters because of the expected approximate linear relation expected between discharge and width within the practical range of values for width. A fit of the data may be easily made using log-linear least square regression. Note that with this formulation, some of the constants may be found to have negative values.

The data tables created for this approach consist of 6 columns each. These 6 columns are the terms of equation 1a. Two data tables are created. The first is the data for the 16" headcut width, and the second is the table for the 8" width. The first is used for the determination of the coefficient and exponents of equation 1a, and the second is used as a check on the relation developed.

APPROACH 2: Dimensional Analysis With Mass and Energy Flux Considered

It is also possible to take a more physically based approach to the determination of the nature of the functional relation between head and discharge. For this approach, it is instructive to look at the mass and energy balance for a control volume with the upstream surface well into the reservoir and the downstream surface at the point of maximum contraction near the brink of the headcut as shown below.



Then the basic control volume equations may be applied. The equation describing conservation of mass may be written as:

$$0 = \frac{\partial}{\partial t}\int_{cv}\rho d\varsigma + \int_{cs}\rho\vec{v} \bullet d\vec{A} \qquad (2)$$

where

ρ = fluid density
ς = volume
v = velocity
A = area
t = time

the arrows are used to indicate vector quantities, and the integrations are carried out over the control volume and control surface as indicated.

Assuming

1. *Steady flow conditions (all conditions constant in time),*
2. *incompressible flow (ρ=const.), and*
3. *the control surface to be selected so that surfaces 1 and 2 are everywhere perpendicular to the velocity vector, then equation 2 reduces to the familiar continuity equation of the form:*

$$\int_1 \mathrm{v}dA = \int_2 \mathrm{v}dA = Q \qquad (3)$$

where velocity and area are now scaler quantities, the integrals are carried out over control volume surfaces 1 and 2 respectively, and Q is the volumetric discharge through the control volume (also through the breach).

Conservation of energy may be written as:

$$\frac{\delta Q_h}{\delta t} - \frac{\delta W_s}{\delta t} = \frac{\partial}{\partial t}\int_{cv}\rho e d\varsigma + \int_{cs}(\frac{p}{\rho} + e)\rho\vec{v} \bullet d\vec{A} \qquad (4)$$

in which the terms on the left side of the equation represents heat energy transferred into the control volume and mechanical energy transferred out of the control volume respectively, the internal energy per unit mass of the flowing fluid is given by:

$$e = gz + \frac{\mathrm{v}^2}{2} + u \qquad (5)$$

p = pressure,
g = gravitational acceleration,
z = elevation,

u = the intrinsic energy per unit mass of the fluid,

and the other terms are as previously defined.

For the control volume shown, assuming conditions as described above plus:

4. *No heat transfer or work done on the solid boundary (ie; no frictional losses or forces),*

then all terms of equation 4 are zero except for the integral over the control surface, and that integral may be reduced to a scaler form as:

$$\int_{cs1}\left(\frac{p}{\rho}+gz+\frac{\mathrm{v}^2}{2}+u\right)\mathrm{v}dA=\int_{cs2}\left(\frac{p}{\rho}+gz+\frac{\mathrm{v}^2}{2}+u\right)\mathrm{v}dA \tag{6}$$

further assuming:

5. *the intrinsic energy of the fluid to remain constant throughout the control volume (consistent assumption with no. 4)*

then continuity (equation 3) results in the terms involving intrinsic energy, *u*, to drop out of the equation. Selecting control surface 1 such that it is in the reservoir, it is reasonable to assume:

6. *The velocity through control surface 1 is sufficiently small that the square of the velocity approaches zero (velocity head at 1 negligable),*
7. *The pressure is hydrostatic over surface 1 ($p=\rho g(z_{ws1}-z)$)*

which allows equation 6 to be written as:

$$gz_{ws1}Q=\int_{cs2}\left(\frac{p}{\rho}+gz+\frac{\mathrm{v}^2}{2}\right)\mathrm{v}dA \tag{7}$$

Defining the area, mean velocity and energy coefficient in the usual fashion as:

$$A_2=\int_{cs2}dA \tag{8}$$

$$V_2=\frac{Q}{A_2} \tag{9}$$

$$\alpha=\frac{\int_{cs2}\mathrm{v}^3dA}{V_2^3A_2} \tag{10}$$

reduces equation 7 to the familiar form

$$gz_{ws1}\mathrm{Q} = \int_{cs2}(\frac{p}{\rho} + gz)\mathrm{v}d\mathrm{A} + \alpha\frac{\mathrm{V}_2^2}{2}\mathrm{Q} \tag{11}$$

A pressure deviation term may then be defined as:

$$P_d = (\int_{cs2}\rho g(z_c + \mathrm{d} - z)\mathrm{v}d\mathrm{A} - \rho\int_{cs2}\frac{p}{\rho}\mathrm{v}d\mathrm{A})/\mathrm{Q} \tag{12}$$

where

z_c= elevation of the elevation of the brink or crest of the headcut,
d= an arbitrary value approximately equal to the elevation difference between the water surface and the brink or crest of the headcut,

and the other terms are as previously defined.

Substituting equation 12 into equation 11 yeilds:

$$gz_{ws1}\mathrm{Q} = gz_c\mathrm{Q} + g\mathrm{dQ} - \mathrm{Q}\frac{P_d}{\rho} + \alpha\frac{\mathrm{V}_2^2}{2}\mathrm{Q} \tag{13}$$

or

$$h_e = \mathrm{d} - \frac{P_d}{\rho g} + \alpha\frac{\mathrm{V}_2^2}{2g} \tag{14}$$

Defining a pressure coefficient such that:

$$\alpha_p = \frac{P_d\mathrm{d}}{\int_0^d \rho g y dy} = \frac{P_d}{\rho g \frac{\mathrm{d}}{2}} \tag{15}$$

allowing equation 14 to be written as

$$h_e = (1 - \frac{\alpha_p}{2})\mathrm{d} + \alpha\frac{\mathrm{Q}^2}{2\mathrm{gA}_2^2} \tag{16}$$

Recognizing d as an <u>approximation</u> of the depth at the overfall brink, it is possible to define a form of a contraction coefficient as:

$$C_c = \frac{\mathrm{A}_2}{(b + \mathrm{d}\tan(\phi))\mathrm{d}} \tag{17}$$

and, recognizing that the flow is passing through critical in the vicinity of section 2, it is possible to define an additional coefficient as:

$$C_s = \frac{d}{h_e} \tag{18}$$

substituting into equation 16 yeilds:

$$h_e = \left(1 - \frac{\alpha_p}{2}\right) C_s h_e + \frac{\alpha Q^2}{2g[C_c(b + C_s h_e \tan(\phi))C_s h_e]^2} \tag{19}$$

which expresses the head discharge relation in terms of four coefficients that are functions of the boundary geometry. Rearranging to express Q in terms of h_e yeilds:

$$Q = \left\{ \frac{2g}{\alpha} [C_c C_s h_e (b + C_s h_e \tan(\phi))]^2 h_e \left[1 - C_s \left(1 - \frac{\alpha_p}{2}\right)\right]\right\}^{\frac{1}{2}} \tag{20}$$

The advantage of this relation is that each of the coefficients has a physical meaning that allows it to be bounded and rational evaluation to be made of geometric influences on it. The disadvantage of this formulation is that it is cumbersome and is not suitable for use in other than machine calculations.

Looking first at the pressure coefficient, α_p, it may be noted that it represents the deviation from hydrostatic pressure over surface 2. For the condition of headcut depth, h_h, of zero, it may be reasonably assumed that the velocity is horizontal, pressure is hydrostatic, and the depth term is the actual flow depth. From equation 12, then the two integrals are identical, the pressure deviation term, P_d, is equal to zero, and α_p is equal to zero.

For a condition of headcut height sufficient to for the nappe to break free and form a jet, surface 2 may be selected such that pressure everywhere on the surface is equal to zero. In this case, the pressure deviation term becomes the first integral of equation 12, which is a bit more difficult to evaluate in general terms. The order of magnitude of the term may be evaluated, however, by making the approximating assumptions that a)the velocity is horizontal at section 2, b) the velocity is constant through the section, and c) the width of the section is constant (ie: ϕ=0). Under these simplifying assumptions, the integration may be carried out to yield $P_d = \rho g d/2$ or α_p=1. Therefore, this coefficient would be expected to range from zero to approximately 1, to be dominated by the relative height of the headcut, h_h/h_e, with secondary dependence on the other geometric parameters, principally relative width and breach side slope.

Taking the remaining parameters individually: α_p would be expected to increase with increasing θ, with only a weak relation expected. Dependence on ϕ would also be expected to be slight with an inverse relation anticipated. The upstream apron height would also be anticipated to have only a slight influence with increasing h_u slightly increasing α_p. The influence of base width is not entirely clear. The relative impact of the bed width on the influence of breach side slope, ϕ, may be greater than any other impact, causing it to appear to have a different influence than would otherwise be the case. Neglecting this interaction, one would expect that narrowing the base width would tend to increase α_p by tending to decrease the pressure due to streamline curvature.

The surface contraction coefficient, C_s may also be computed for idealized conditions, although it is more difficult to determine which of the geometric parameters will dominate its variation. From inspection, it may be determined that this coefficient will always be greater than zero and less than 1. That is, there must be some depth of flow over the headcut when there is head on the system, and the water surface at section 2 will never be greater than the water surface at section 1. For the simplified condition of a section of unit width through a wide breach, uniform velocity, and hydrostatic pressure, and critical depth at section 2, it may be shown that C_s=0.67 (critical depth is equal to 2/3 of the head). Pressure decreases at section 2 would tend to cause C_s to increase, whereas energy losses (assumed zero in the derivation) would tend to cause it to decrease.

Taking the variables in the same order as before, it is possible to crudely estimate the nature of the relation of C_s to each. An increase in headcut height would correspond to a decrease in pressure at section 2, and therefore tend to increase C_s. Again, only a weak relation with approach slope, θ, would be expected, and that too would be related to pressure, with the increasing θ decreasing pressure and increasing C_s. Dependence on breach side slope, ϕ, would also be expected to be weak, with increasing ϕ causing and increase in C_s. The relative elevation of the upstream bed would again be related to pressure, with increases in h_u resulting in a decrease in pressure and an increase in C_s. The effect of bed width is again unclear, but anticipated to be inversely related.

The contraction coefficient, C_c, is the traditional contraction coefficient describing the reduction in area associated with flow orifice type flow. It is a function of the boundary geometry dominated by the relative size of the opening. As formulated, it would be bounded by 0 and 1.0 and would be primarily a function of relative width and breach side slope, with secondary dependence on the other geometric variables. Typical values would be expected to be on the order of 0.8.

Influence of headcut height, h_h, on C_c would be expected to be related primarily to the shape of the underside of the nappe. Therefore, C_c, would be expected to decrease with increasing h_h. The influence of approach slope, θ, would also be slight and associated with the nappe shape, with C_c being expected to decrease with increasing θ. As indicated above, the influence of breach side slope would be through its effect on the relative size of the breach. Therefore, increasing ϕ would be expected to increase C_c through reducing the extent of contraction. The relative elevation of the upstream bed would again be expected to have only a minor influence through its influence on the nappe shape, with increasing h_u resulting in a decrease in C_c. The headcut base width, b, would be expected to be the dominant parameter, with increasing bed width causing an increase in C_c (very wide headcut, no contraction, C_c equal to 1.0).

Finally, the coefficient, α, is the traditional energy coefficient representing the variation in velocity passing through control surface 2. It has a lower bound of 1.0, and a typical value lying between 1.0 and 1.1. Its value would be expected to be dominated by the presence or absence of the influence of a lower boundary at 2, and therefore, would dominated by the variable h_h.

Since deviation of the local velocity from the mean velocity through the section causes α to increase, it is expected that α would be inversely related to headcut height, h_h. The influence of the approach slope would again be small, with α expected to decrease with increasing θ. The effect of the breach side slope would again be related to the relative size of the opening with α expected to decrease with increasing ϕ. The elevation of the upstream bed would affect the approach velocity profile, with α being expected to decrease with increasing h_u. α would also be expected to decrease with increasing relative width of headcut.

PARAMETRIC FORMULATION

Based on the above discussion, it is possible to formulate relations with the behavior that would be anticipated for each of the coefficients identified. It should be noted that the forms developed below are not the only forms that could be applied, but rather are selected to be as simple as possible while retaining the expected behavior as described above. It is worth noting that the analysis to be carried out did not require that the nature of the influence of each parameter be determined in advance. However, by doing so, the problem may be formulated such that the anticipated value for each of the coefficients and exponents is positive. This provides an additional check on the logic used in formulating the parameters.

As formulated below, the assumption of independent influence of each of the 5 geometric variables, h_h, θ, h_u, b, and ϕ, is implied. Actually, this cannot be the case. For example, the approach slope, θ, and the elevation of the upstream bed, h_u, will necessarily be interrelated in their impact on the magnitude and direction of the velocity near the base of the headcut. However, it is not considered appropriate to try to sort out this type of interaction in the preliminary analysis, because it will likely be found that the impact of some of the geometric variables will not be significant for all parameters. That is, some of the exponents in the following relations will be found to be effectively zero.

Following the same order as above, and beginning with the coefficient α_p, we obtain the form.

$$\alpha_p = c_1\left(\frac{h_h}{h_e + h_h}\right)^{c_2}\left(\frac{2\theta}{\pi}\right)^{c_3}\left(\frac{h_u + h_e}{2h_e + h_u}\right)^{c_4}\left(\frac{b + h_e}{2b + h_e}\right)^{c_5}\left(\frac{\pi - \phi}{\pi}\right)^{c_6} \qquad (21)$$

4th?

with the formulation of the terms again being somewhat arbitrary, but consistent with anticipated behavior. Note that in this initial formulation, the first term is allowed to go to zero with a headcut height of zero, forcing the remaining terms to have no influence on α_p for this condition. Note also that the 5[th] term is formulated such that a positive exponent would imply that increasing base headcut width would decrease α_p. Further examination following initial analysis may suggest a different formulation. The beginning values for gradient search analysis given as:

$c_1 = 1.0$
$c_2 = 0.3$
$c_3 = 0.05$
$c_4 = 0.05$
$c_5 = 0.05$
$c_6 = 0.05$

Note that the beginning values are consistent with the idea that the action will be dominated by the first term.

The surface contraction coefficient, C_s, may be formulated in similar terms as:

$$C_s = c_7\left(\frac{h_h + h_e}{h_h + 2h_e}\right)^{c_8}\left(\frac{2\theta}{\pi}\right)^{c_9}\left(\frac{h_u + h_e}{2h_e + h_u}\right)^{c_{10}}\left(\frac{b + h_e}{2b + h_e}\right)^{c_{11}}\left(\frac{\pi + \phi}{2\pi + \phi}\right)^{c_{12}} \qquad (22)$$

with beginning points for gradient search analysis given as:

$c_7 = 0.67$
$c_8 = 0.2$
$c_9 = 0.05$
$c_{10} = 0.05$
$c_{11} = 0.05$
$c_{12} = 0.05$

Again the first term is expected to dominate as indicated by the starting values.

The contraction coefficient, C_c, may be formulated in similar fashion as:

$$C_c = c_{13}\left(\frac{h_h + h_e}{2h_h + h_e}\right)^{c_{14}}\left(\frac{\pi + \theta}{\pi + 2\theta}\right)^{c_{15}}\left(\frac{h_u + h_e}{2h_u + h_e}\right)^{c_{16}}\left(\frac{b + h_e}{b + 2h_e}\right)^{c_{17}}\left(\frac{\pi + \phi}{2\pi - \phi}\right)^{c_{18}} \qquad (23)$$

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with beginning points and initial step sizes for gradient search analysis given as:

$c_{13} = 1.0$
$c_{14} = 0.05$
$c_{15} = 0.05$

c_{16}= 0.05
c_{17}= 0.5
c_{18}= 0.1

consistent with the anticipation that the last two terms would dominate the relation.

And, the energy coefficient, α, may be formulated as,

$$\alpha = c_{19}\left(\frac{h_h + h_e}{2h_h + h_e}\right)^{c_{20}}\left(\frac{\pi+\theta}{\pi+2\theta}\right)^{c_{21}}\left(\frac{h_u + h_e}{2h_u + h_e}\right)^{c_{22}}\left(\frac{b + h_e}{2b + h_e}\right)^{c_{23}}\left(\frac{\pi-\phi}{\pi}\right)^{c_{24}} \tag{24}$$

with beginning points for gradient search analysis given as:

c_{19} = 1.1
c_{20} = 0.2
c_{21} = 0.05
c_{22}= 0.1
c_{23}= 0.05
c_{24}= 0.05

The problem is then formulated in terms 4 physically based, variable coefficients, each of which is a function of 5 dimensionless variables and 6 constants for a total of 24 empirical constants. Since there are approximately 2500 data points from which to fit the constants, the formulation is manageable. However, as noted above, the influence of some of the dimensionless variables on some of the variable coefficients may be found not to be significant.

Data Table Form

Examination of the above relation shows that the seven quantities represented in the raw data are represented in various ways. To reduce computations, the data table is set up with a total of 18 columns. The first 7 columns are the raw data, and the next 11 columns are dimensionless forms. As with approach 1, two data tables are to be created. One for the 16" headcut width and one for the 8". The order of the first seven variables and of the last 11 variables may be changed as long as consistency is maintained between the tables, and the order is made clear. Development of a routine to shift column order is not difficult. Suggested order is:

Q, h_e, h_h, θ, h_u, b, φ

$$\left(\frac{h_h}{h_h + h_e}\right), \left(\frac{h_h + h_e}{h_h + 2h_e}\right), \left(\frac{h_h + h_e}{2h_h + h_e}\right),$$

$$\left(\frac{2\theta}{\pi}\right), \left(\frac{\pi+\theta}{\pi+2\theta}\right),$$

$$\left(\frac{h_u + h_e}{h_u + 2h_e}\right), \left(\frac{h_u + h_e}{2h_u + h_e}\right),$$

$$\left(\frac{b + h_e}{b + 2h_e}\right), \left(\frac{b + h_e}{2b + h_e}\right),$$

$$\left(\frac{\pi + \phi}{2\pi + \phi}\right), \left(\frac{\pi - \phi}{\pi}\right)$$

All lengths are in feet, volumes in cubic feet, and angles in radians.

Setting the data table up in this fashion represents a logical grouping of the variables of manipulation for input into an analysis routine for the purpose of performing a gradient search for optimum values of the various coefficients and exponents. After examination of the data fit and of the statistical significance of the various parameters, it may be appropriate to revisit the overall problem formulation.